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Highest weight modules over $W_{1+\infty}$ algebra and the bispectral problem

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0 Introduction

This paper is the last of a series of papers devoted to the bispectral problem [3]–[6]. Here we examine the connection between the bispectral operators constructed in [6] and the Lie algebra $W_{1+\infty}$ (and its subalgebras). To give a more detailed idea of the contents of the present paper we briefly recall the results of [4]–[6] which we need.

In [4] we built large families of representations of $W_{1+\infty}$. For each $\beta \in \mathbb{C}^N$ we defined a tau-function $\tau_\beta(t)$ which we called *Bessel tau-function*. We proved that it is a highest weight vector for a representation $\mathcal{M}_\beta$ of the algebra $W_{1+\infty}$ with central charge N. In [6] we introduced a version of Darboux transformation, which we called *monomial*, on the corresponding wave functions $\Psi_\beta(x,z)$ (see also Subsect. 1.2) and showed that the resulting wave functions are bispectral. For example all bispectral operators from [9, 22] can be obtained in this way.

The present paper establishes closer connections between $W_{1+\infty}$ and the bispectral problem. Our first result (Theorem 2.1) shows that a tau-function is a monomial Darboux transformation of a Bessel tau-function if and only if it belongs to one of the modules $\mathcal{M}_\beta$. This type of connection between the representation theory (of $W_{1+\infty}$) and the bispectral problem is, to the best of our knowledge, new even for the bispectral tau-functions of Duistermaat and Grünbaum [9].

The second of the questions we try to answer in the present paper originates from Duistermaat and Grünbaum [9]. They noticed that their rank 1 bispectral operators are invariant under the KdV-flows and asked if there is a hierarchy of symmetries for the rank 2 bispectral operators. The latter question was answered affirmatively by Magri and Zubelli [17] who showed that the algebra Vir^+ (the subalgebra of the Virasoro algebra spanned by the operators of non-negative weight) is tangent to

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the manifold of rank 2 bispectral operators. Here we obtain generalizations of these results as follows.

First we show that the flows generated by $W^+_{1+\infty}(N)$ leave our manifold of monomial Darboux transformations $Gr^{(N)}_{MB}$ invariant (see Theorem 3.1). An important feature of our proof is that it naturally follows from the results contained in Theorem 2.1 about the tau-functions in the modules $\mathcal{M}_\beta$. For this reason we believe that even in the case $N = 2$ [17] it gives a better explanation of the origin of the flows. Note that the corresponding bispectral operators need not to be of order N as in [17]. Our next result touches upon this particular situation. We consider the manifolds of rank N polynomial Darboux transformations (see Subsect. 1.2) of Bessel tau-functions for which the spectral algebra contains an operator of order N. Then a natural bosonic realization of Vir^+_N generates flows leaving such manifolds invariant (see Theorem 3.3). For $N = 2$ our theorem coincides with cited above result of [17].

The monomial Darboux transformations form a subfamily of a larger class of solutions to the bispectral problem – the polynomial Darboux transformations of Bessel and Airy planes [6]. It is a very interesting open problem to find hierarchies of symmetries preserving the manifolds of polynomial Darboux transformations. We think that this problem is also connected to representation theory. Perhaps the vertex operator algebra structure of $W_{1+\infty}$ [11] and of (certain completions of) the modules $\mathcal{M}_\beta$ will help in tackling this question.

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1 Preliminaries

Here we have collected some facts and notation needed in Sect. 2 and 3. Most proofs are standard but technical and are given in the Appendix.

1.1

In this subsection we recall some facts and notation from Sato's theory of KP-hierarchy [18, 8, 19] needed in the paper. We use the approach of V. Kac and D. Peterson based on infinite wedge products (see e.g. [16]) and the recent survey paper by P. van Moerbeke [20].

Consider the infinite-dimensional vector space of formal series

$$\mathbb{V} = \Big\{ \sum_{k\in\mathbb{Z}} a_k v_k \Big| \, a_k = 0 \text{ for } k \ll 0 \Big\}.$$

Then define the fermionic Fock space F to be the direct sum of the spaces $F^{(m)}$ (states with a charge number m) consisting of formal infinite sums of semi-infinite wedge monomials

$$v_{i_0} \wedge v_{i_1} \wedge \dots$$

such that $i_0 > i_1 > \ldots$ and $i_k = m - k$ for $k \gg 0$.

There exists a well known linear isomorphism, called a *boson-fermion correspondence*:

$$\sigma \colon F \to B := \mathbb{C}\left[\left[t_1, t_2, \ldots; Q, Q^{-1}\right]\right] \tag{1.1}$$

(see [16] and the Appendix).

Sato's Grassmannian Gr [18, 8, 19] consists of all subspaces $W \subset \mathbb{V}$ which have an admissible basis

$$w_k = v_k + \sum_{i>k} w_{ik} v_i, \quad k = 0, -1, -2, \ldots$$

To a plane $W \in Gr$ we associate a state $|W\rangle \in F^{(0)}$ as follows

$$|W\rangle = w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots$$

A change of the admissible basis results in a multiplication of $|W\rangle$ by a non-zero constant. Thus we define an embedding of Gr into the projectivization of $F^{(0)}$ which is called a Plücker embedding. One of the main objects of Sato's theory is the *tau-function* of W defined as the image of $|W\rangle$ under the boson-fermion correspondence (1.1)

$$\tau_W(t) = \sigma(|W\rangle) = \sigma(w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots). \tag{1.2}$$

It is a formal power series in the variables $t_1, t_2, \ldots$, i.e. an element of $B^{(0)} := \mathbb{C}\left[\left[t_1, t_2, \ldots\right]\right]$. Another important function connected to W is the *Baker* or *wave function*

$$\Psi_W(t, z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau\left(t - [z^{-1}]\right)}{\tau(t)}, \tag{1.3}$$

where $[z^{-1}]$ is the vector $\left(z^{-1}, z^{-2}/2, \ldots\right)$. Most often Ψ_W is viewed as a formal series. Introducing the vertex operator

$$X(t, z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k z^k} \frac{\partial}{\partial t_k}\right) \tag{1.4}$$

the above formula (1.3) can be written as

$$\Psi_W(t, z) = \frac{X(t, z)\tau(t)}{\tau(t)}. \tag{1.5}$$

We often use the formal series $\Psi_W(x, z) = \Psi_W(t, z)|_{t_1 = x, t_2 = t_3 = \cdots = 0}$, which we call again a Baker function. The Baker function $\Psi(x, z)$ contains the whole information about W and hence about τ_W, as the vectors $w_{-k} = \partial_x^k \Psi_W(x, z)|_{x=0}$ form an admissible basis of W (if we take $v_k = z^{-k}$ as a basis of $\mathbb{V}$).

We also use the standard notation

$$Gr^{(N)} = \{V \in Gr | z^N V \subset V\}.$$

For $V \in Gr^{(N)}$ there exists an operator $L_V(x, \partial_x)$ of order N such that

$$L_V(x, \partial_x) \Psi_V(x, z) = z^N \Psi_V(x, z)$$

and the corresponding tau-function $\tau_V(t)$ does not depend on $t_N, t_{2N}, \ldots$

1.2

Here we shall briefly recall the definition of Bessel wave function and of monomial Darboux transformations from it. For more details see [6].

Let $\beta \in \mathbb{C}^N$ be such that

$$\sum_{i=1}^{N} \beta_i = \frac{N(N-1)}{2}. \tag{1.6}$$

Definition 1.1 [10, 22, 4] *Bessel wave function* is called the unique wave function $\Psi_\beta(x,z)$ depending only on xz and satisfying

$$L_\beta(x,\partial_x)\Psi_\beta(x,z) = z^N \Psi_\beta(x,z), \tag{1.7}$$

where

$$L_\beta(x,\partial_x) = x^{-N}(D_x-\beta_1)(D_x-\beta_2)\cdots(D_x-\beta_N), \tag{1.8}$$

which is called a *Bessel operator* ($D_x = x\partial_x$). The corresponding plane $V_\beta \in Gr$ in Sato's Grassmannian is called a *Bessel plane* (it has an admissible basis $w_{-k} = \partial_x^k \Psi_\beta(x,z)|_{x=1}$ if we take $v_k = e^z z^{-k}$ as a basis of $\mathbb{V}$).

Remark 1.2 In the above definition we use a convention from [6] which we shall recall. For a plane $W \in Gr$ such that $\Psi_W(x,z)$ is well defined for $x = x_0$ we set $v_k = e^{x_0 z} z^k$ and consider the subspace W^{x_0} of $\mathbb{V}$ with an admissible basis $w_k = \partial_x^k \Psi_W(x,z)|_{x=x_0}$. The wave functions of W^{x_0} and W are connected by $\Psi_{W^{x_0}}(x,z) = e^{-x_0 z}\Psi_W(x+x_0,z)$ and obviously

$$\tau_W(t_1,t_2,t_3,\dots) = \tau_{W^{x_0}}(t_1-x_0,t_2,t_3,\dots)$$

where the RHS is considered as a formal power series in $t_1-x_0, t_2, t_3, \dots$

Throughout the paper we work with the spaces W^1 without explicitly mentioning it and our tau-functions are formal power series in $t_1-1, t_2, t_3, \dots$ $\square$

Because the Bessel wave function depends only on xz, (1.7) implies

$$D_x\Psi_\beta(x,z) = D_z\Psi_\beta(x,z), \tag{1.9}$$

$$L_\beta(z,\partial_z)\Psi_\beta(x,z) = x^N\Psi_\beta(x,z). \tag{1.10}$$

The monomial Darboux transformations of Bessel wave functions were introduced in our previous paper [6]. They are a part of the solutions to the bispectral problem (polynomial Darboux transformations) which we constructed there.

First recall the definition of polynomial Darboux transformations given in [6].

Definition 1.3 We say that a plane W (or the corresponding wave function $\Psi_W(x,z)$) is a *Darboux transformation* of the Bessel plane V_β (respectively wave function

$\Psi_\beta(x,z)$) iff there exist polynomials $f(z)$, $g(z)$ and differential operators $P(x,\partial_x)$, $Q(x,\partial_x)$ such that

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Psi_\beta(x,z), \tag{1.11}$$

$$\Psi_\beta(x,z) = \frac{1}{f(z)} Q(x,\partial_x)\Psi_W(x,z). \tag{1.12}$$

The Darboux transformation is called *polynomial* iff the operator $P(x,\partial_x)$ from (1.11) has the form

$$P(x,\partial_x) = x^{-n}\sum_{k=0}^{n} p_k(x^N) D_x^k, \tag{1.13}$$

where p_k are rational functions, $p_n \equiv 1$.

(In [6] we normalized $g(z)$. For the present paper the normalization is unnecessary.)

There are two equivalent definitions of monomial Darboux transformations of Bessel wave functions (see [6]).

Definition 1.4 We say that the wave function $\Psi_W(x,z)$ (or the corresponding plane W) is a *monomial Darboux transformation* of the Bessel wave function $\Psi_\beta(x,z)$ (respectively the plane V_β) iff it is a polynomial Darboux transformation of $\Psi_\beta(x,z)$ with $g(z)f(z) = z^{dN}$, $d \in \mathbb{N}$.

Definition 1.5 The wave function $\Psi_W(x,z)$ (or the corresponding plane W) is a *monomial Darboux transformation* of the Bessel wave function $\Psi_\beta(x,z)$ (respectively the plane V_β) iff (1.11) holds with $g(z) = z^n$, $n = \mathrm{ord}P$ and the kernel of the operator $P(x,\partial_x)$ has a basis consisting of several groups of the form

$$\partial_y^l \Big(\sum_{k=0}^{k_0} \sum_{j=0}^{\mathrm{mult}(\beta_i+kN)-1} b_{kj} x^{\beta_i+kN} y^j \Big)\Big|_{y=\ln x}, \quad 0 \le l \le j_0, \tag{1.14}$$

where $\mathrm{mult}(\beta_i + kN) :=$ multiplicity of $\beta_i + kN$ in $\bigcup_{j=1}^{N}\{\beta_j + N\mathbb{Z}_{\ge 0}\}$ and $j_0 = \max\{j | b_{kj} \ne 0$ for some $k\}$.

Denote the set of monomial Darboux transformations of V_β by $Gr_{MB}(\beta)$. For polynomial Darboux transformations we use the notation $Gr_B(\beta)$.

A simple consequence of the above definitions is that

$$Q(x,\partial_x)P(x,\partial_x) = L_\beta(x,\partial_x)^d, \tag{1.15}$$

where

$$\mathrm{ord}P = n,\ \mathrm{ord}Q = dN - n \quad (g(z) = z^n,\ f(z) = z^{dN-n}). \tag{1.16}$$

Note that the monomial Darboux transformations have the following transitivity and reflexivity properties:

$$W \in Gr_{MB}(\beta),\ V_\beta \in Gr_{MB}(\beta') \ \Rightarrow\ W \in Gr_{MB}(\beta');$$
$$V_\beta \in Gr_{MB}(\beta') \ \Leftrightarrow\ V_{\beta'} \in Gr_{MB}(\beta).$$

1.3

In this subsection we recall the definition of $W_{1+\infty}$, its subalgebras $W_{1+\infty}(N)$ and their bosonic representations introduced in [4].

The algebra w_∞ of the additional symmetries of the KP–hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle

$$\mathcal{D} = \mathrm{span}\{z^\alpha \partial_z^\beta |\ \alpha, \beta \in \mathbb{Z},\ \beta \geq 0\}.$$

Its unique central extension [14, 15] will be denoted by $W_{1+\infty}$. This algebra gives the action of the additional symmetries on tau-functions (see [1]). Denote by c the central element of $W_{1+\infty}$ and by $W(A)$ the image of $A \in \mathcal{D}$ under the natural embedding $\mathcal{D} \hookrightarrow W_{1+\infty}$ (as vector spaces). The algebra $W_{1+\infty}$ has a basis

$$c,\ J_k^l = W(-z^{l+k}\partial_z^l), \qquad l, k \in \mathbb{Z},\ l \geq 0.$$

The commutation relations of $W_{1+\infty}$ can be written most conveniently in terms of generating series [15]

$$\Big[W(z^k e^{xD_z}), W(z^m e^{yD_z})\Big] = (e^{xm} - e^{yk}) W(z^{k+m} e^{(x+y)D_z}) + \delta_{k,-m} \frac{e^{xm} - e^{yk}}{1 - e^{x+y}} c, \tag{1.17}$$

where $D_z = z\partial_z$.

Instead of working with the generators J_k^l it is much more convenient to work with the generating functions or fields (of dimension $l+1$)

$$J^l(z) = \sum_{k \in \mathbb{Z}} J_k^l z^{-k-l-1}. \tag{1.18}$$

The modes $J_k = J_k^0$ of the $\hat{u}(1)$ current $J(z) = J^0(z)$ generate the Heisenberg algebra:

$$\Big[J_n, J_m\Big] = n\delta_{n,-m} c. \tag{1.19}$$

Recall its canonical representation in the bosonic Fock space B:

$$J_n = \frac{\partial}{\partial t_n}, \quad J_{-n} = n t_n, \quad n > 0, \quad J_0 = Q\frac{\partial}{\partial Q}, \quad c = 1. \tag{1.20}$$

It is well known that for $c = 1$ the fields $J^l(z)$ can be expressed as normally ordered polynomials in the current $J(z)$:

$$J^l(z) = l!\, {:} S_{l+1}\Big(\frac{J(z)}{1!}, \frac{\partial J(z)}{2!}, \dots\Big){:}. \tag{1.21}$$

Here as usual

$$:J_n J_m: = \begin{cases} J_n J_m & \text{for} \quad m > n \\ J_m J_n & \text{for} \quad m < n \end{cases}$$

and the elementary Schur polynomials $S_l(t)$ are determined by the generating series

$$\exp\Big(\sum_{k=1}^{\infty} t_k z^k\Big) = \sum_{l=0}^{\infty} S_l(t) z^l. \tag{1.22}$$

Substituting (1.20) in (1.21) we obtain a bosonic representation of $W_{1+\infty}$ with central charge $c = 1$. For an explanation and a proof of (1.21) see the Appendix.

In [4] we constructed a family of highest weight modules of $W_{1+\infty}$ using the above bosonic representation. We shall sum up the results from that paper in a suitable for our purposes form. First we introduce the subalgebra $W_{1+\infty}(N)$ of $W_{1+\infty}$ spanned by c and J^l_{kN}, $l, k \in \mathbb{Z}$, $l \geq 0$. It is a simple fact that $W_{1+\infty}(N)$ is isomorphic to $W_{1+\infty}$ (see [12]).

Theorem 1.6 *The functions $\tau_\beta(t)$ satisfy the constraints*

$$J^l_0 \tau_\beta = \lambda_\beta(J^l_0)\tau_\beta, \quad l \geq 0, \tag{1.23}$$

$$J^l_{kN}\tau_\beta = 0, \quad k > 0,\ l \geq 0, \tag{1.24}$$

$$W\Big(z^{-kN} P_{\beta,k}(D_z) D^l_z\Big)\tau_\beta = 0, \quad k > 0,\ l \geq 0, \tag{1.25}$$

where $P_{\beta,k}(D_z) = P_\beta(D_z)P_\beta(D_z - N)\cdots P_\beta(D_z - N(k-1))$ and $P_\beta(D_z) = (D_z - \beta_1)\cdots(D_z - \beta_N)$.

The first two constraints mean that τ_β is a highest weight vector with highest weight λ_β of a representation of $W_{1+\infty}(N)$ in the module

$$\mathcal{M}_\beta = \mathrm{span}\Big\{ J^{l_1}_{k_1 N} \cdots J^{l_p}_{k_p N}\tau_\beta \Big| k_1 \leq \ldots \leq k_p < 0 \Big\}. \tag{1.26}$$

In [4] we studied $\mathcal{M}_\beta$ as modules of $W_{1+\infty}$. We proved that they are *quasifinite* (see [15]) and we derived formulae for the highest weights and for the singular vectors. The latter formula turns out to be the simplest corollary of Theorem 2.6 (see Example 2.10).

1.4

In the next sections we shall need the action of the so-called *adjoint involution a* on the modules $\mathcal{M}_\beta$. On the tau-functions it acts as follows [21]:

$$\tau_{aV}(t_1, t_2, \ldots, t_k, \ldots) = \tau_V(t_1, -t_2, \ldots, (-1)^{k-1}t_k, \ldots). \tag{1.27}$$

We continue this action on $B^{(0)} = \mathbb{C}\,[[t_1 - 1, t_2, t_3, \ldots]]$ (cf. Remark 1.2).

We shall continue it also on the elements of $W_{1+\infty}$ in its bosonic representation (1.21) naturally demanding

$$a(U\tau) = a(U)a(\tau) \ \text{ for } U \in W_{1+\infty},\ \tau \in B^{(0)}.$$

It acts on the Heisenberg algebra by $a(J_k) = (-1)^{k-1}J_k$, i.e. $a(J(z)) = J(-z)$, and on the fields $J^l(z)$ via (1.21).

Proposition 1.7 *If $\tau \in \mathcal{M}_\beta$ ($\beta \in \mathbb{C}^N$) then $a(\tau) \in \mathcal{M}_{a(\beta)}$ where $a(\beta) = (N-1)\delta - \beta$, $\delta = (1, 1, \ldots, 1)$.*

Proof. Using the commutation relations (1.17) one can prove by induction on the dimension of the fields $J^l(z)$ that a preserves $W_{1+\infty}(N)$ (as a basis of the induction one uses (1.21) for $l = 0, 1, 2$). In the Appendix we give another proof of this fact providing an explicit expression for a basis of $W_{1+\infty}$ in which a acts diagonally. The proposition now follows from the fact that $a(\tau_\beta) = \tau_{a(\beta)}$ (see [6]). $\square$

2 Tau-functions in Bessel modules as monomial Darboux transformations

This section examines the connection between the class of representations obtained in [4] (see Subsect. 1.3) and a part of the solutions to the bispectral problem constructed in [6] (see Subsect. 1.2). Our main result is the following.

Theorem 2.1 *If τ_W is a tau-function lying in the $W_{1+\infty}(N)$-module $\mathcal{M}_\beta$ ($\beta \in \mathbb{C}^N$) then the corresponding plane $W \in Gr_{MB}(\beta)$. Conversely, if $W \in Gr_{MB}(\beta)$ then $\tau_W \in \mathcal{M}_{\beta'}$ for some $\beta' \in \mathbb{C}^N$ such that $V_{\beta'} \in Gr_{MB}(\beta)$.*

In general $\beta' \neq \beta$. A more precise version of the second part of the theorem is given in Theorems 2.6 and 2.9 below.

2.1

For the *proof* of the first part of Theorem 2.1 we shall need two lemmas.

Lemma 2.2 *If $\tau \in \mathcal{M}_\beta$ then $\tau = u \cdot \tau_\beta$ with u of the form*

$$u = \sum a_k^l J_{-Nk_1}^{l_1} \cdots J_{-Nk_r}^{l_r}, \tag{2.1}$$

such that all $l_i < Nk_i$.

Proof. For $w = W(z^k P(D_z))$ set $\rho(w) = \mathrm{ord} P + k$. Because of Theorem 1.6 for each $w \in W_{1+\infty}(N)$ there exists $\widetilde{w} \in W_{1+\infty}(N)$ such that $w\tau_\beta = \widetilde{w}\tau_\beta$ and $\rho(\widetilde{w}) < 0$. Then for $w_1, \ldots, w_r \in W_{1+\infty}(N)$ we prove by induction on r that $w_1 \cdots w_r \tau_\beta$ is a sum of elements of the form $\widetilde{w}_1 \cdots \widetilde{w}_s \tau_\beta$ with $\rho(\widetilde{w}_i) < 0$, $s \leq r$. Indeed, for $w \in W_{1+\infty}(N)$

$$\begin{aligned} w\widetilde{w}_1 \cdots \widetilde{w}_s \tau_\beta &= \widetilde{w}_1 \cdots \widetilde{w}_s w \tau_\beta + [w, \widetilde{w}_1 \cdots \widetilde{w}_s]\, \tau_\beta \\ &= \widetilde{w}_1 \cdots \widetilde{w}_s \widetilde{w} \tau_\beta + \sum_{i=1}^{s} \widetilde{w}_1 \cdots [w, \widetilde{w}_i] \cdots \widetilde{w}_s \tau_\beta \end{aligned}$$

with $\rho(\widetilde{w}) < 0$. □

Lemma 2.3 *Let $X(t,z)$ be the vertex operator* (1.4). *Then*

$$X(t,z) J_k^l = \left(J_k^l + l J_k^{l-1} + \delta_{l,0}\delta_{k,0} - z^{k+l}\partial_z^l \right) X(t,z).$$

The *proof* of Lemma 2.3 is given in the Appendix.

Now we can give the *proof* of the first part of Theorem 2.1. Let $\tau_W = u\tau_\beta$ be a tau-function and u be an element of the universal enveloping algebra of $W_{1+\infty}(N)$ of the form (2.1). We compute the wave function

$$\Psi_W(x,z) = \frac{X(t,z)\tau_W(t)}{\tau_W(t)}\Big|_{t_1=x,\, t_2=t_3=\cdots=0}.$$

Using Lemma 2.3 we commute $X(t,z)$ and u to obtain

$$\Psi_W(x,z) = \frac{U(t,z)X(t,z)\tau_\beta(t)}{u\tau_\beta(t)}\Big|_{t_1=x,\, t_2=t_3=\cdots=0}.$$

where

$$U(t,z) = \sum a_k^l \left(J^{l_1}_{-Nk_1} + l_1 J^{l_1-1}_{-Nk_1} - z^{-Nk_1+l_1}\partial_z^{l_1}\right)\cdots$$
$$\cdots\left(J^{l_r}_{-Nk_r} + l_r J^{l_r-1}_{-Nk_r} - z^{-Nk_r+l_r}\partial_z^{l_r}\right).$$

From the bosonic formula (1.21) and the gradation of $W_{1+\infty}(N)$ it is clear that

$$J^l_{-Nk}|_{t_1=x,\, t_2=t_3=\cdots=0} = x^{Nk}\delta_{l+1,Nk} \quad \text{if} \quad l < Nk.$$

What is relevant for us is that there are no differentiations in $t_1, t_2, \ldots$ but only a multiplication by powers of x^N. This gives for $U(t,z)$ the representation

$$U(t,z)|_{t_1=x,\, t_2=t_3=\cdots=0} = \sum a_k^l \left(x^{Nk_1}(\delta_{l_1+1,Nk_1} + l_1\delta_{l_1,Nk_1}) - z^{-Nk_1+l_1}\partial_z^{l_1}\right)\cdots$$
$$\cdots\left(x^{Nk_r}(\delta_{l_r+1,Nk_r} + l_r\delta_{l_r,Nk_r}) - z^{-Nk_r+l_r}\partial_z^{l_r}\right) = z^{-mN}P(x^N, z^N, D_z),$$

for some $m \in \mathbb{N}$ and a polynomial P in x^N, z^N and D_z.

In the same way $u|_{t_1=x,\, t_2=\cdots=0} = g(x^N)$ is polynomial in x^N. Therefore

$$\Psi_W(x,z) = \frac{P(x^N, z^N, D_z)\Psi_\beta(x,z)}{z^{mN}g(x^N)}. \tag{2.2}$$

Using (1.7, 1.9) we obtain

$$\Psi_W(x,z) = z^{-mN}P_1(x^N, D_x)\Psi_\beta(x,z) \tag{2.3}$$

for some operator P_1 with rational coefficients.

We also need an expression for Ψ_β in terms of Ψ_W. It can be obtained by using the adjoint involution a. By Proposition 1.7 $\tau_{aW} = a(u)\tau_{a(\beta)}$ is a tau-function lying in the module $\mathcal{M}_{a(\beta)}$ and (2.3) gives

$$\Psi_{aW}(x,z) = z^{-kN}P_2(x^N, D_x)\Psi_{a(\beta)}(x,z)$$

for some operator P_2. To complete the proof that $W \in Gr_{MB}(\beta)$ we shall apply the following simple lemma (see e.g. [6], Proposition 1.7 (i)).

Lemma 2.4 *If the wave functions $\Psi_W(x,z)$ and $\Psi_V(x,z)$ satisfy*

$$\Psi_W(x,z) = \frac{1}{g(z)}P(x,\partial_x)\Psi_V(x,z),$$

then

$$\Psi_{aV}(x,z) = \frac{1}{\check{g}(z)}P^*(x,\partial_x)\Psi_{aW}(x,z) \tag{2.4}$$

where $\check{g}(z) = g(-z)$ and "" is the formal adjoint (i.e. the antiautomorphism such that $\partial_x^* = -\partial_x$, $x^* = x$).*

The above lemma leads to

$$\Psi_\beta(x,z) = z^{-kN}(-1)^{kN}P_2^*(x^N, D_x)\Psi_W(x,z) \tag{2.5}$$

which combined with (2.3) proves that $W \in Gr_{MB}(\beta)$.

2.2

The second part of Theorem 2.1 is a consequence of Theorems 2.6 and 2.9 below. Before stating the first of them let us introduce some notation and recall some simple facts.

Lemma 2.5 *Let $\beta \in \mathbb{C}^N$ and $\alpha \in \mathbb{C}^M$. Then*

(i) $L_\alpha L_\beta = L_\gamma$, *where*

$$\gamma = (\alpha_1 + N, \alpha_2 + N, \ldots, \alpha_M + N, \beta_1, \beta_2, \ldots, \beta_N);$$

(ii) $(L_\beta)^d = L_{\beta^d}$, *where*

$$\beta^d = (\beta_1, \beta_1 + N, \ldots, \beta_1 + (d-1)N, \ldots, \beta_N, \ldots, \beta_N + (d-1)N);$$

(iii) *If* $\{\beta_1, \ldots, \beta_N\} = \{\underbrace{\alpha_1, \ldots, \alpha_1}_{k_1}, \ldots, \underbrace{\alpha_s, \ldots, \alpha_s}_{k_s}\}$ *with distinct* $\alpha_1, \ldots, \alpha_s$, *then*

$$\mathrm{Ker} L_\beta = \mathrm{span}\left\{x^{\alpha_i}(\ln x)^k\right\}_{1 \le i \le s,\ 0 \le k \le k_i - 1}.$$

The *proof* is obvious.

Let $W \in Gr_{MB}(\beta)$ be a monomial Darboux transformation of the Bessel plane V_β, $\beta \in \mathbb{C}^N$. We can consider only the case when $n \le d$ (see eqs. (1.15, 1.16)) since the general case can be reduced to this one by a left multiplication of Q by L_β, which does not change W and β. Let $\gamma = \beta^d$ (see Lemma 2.5 (ii)), i.e.

$$\gamma_{(k-1)d+j} := \beta_k + (j-1)N, \quad 1 \le k \le N,\ 1 \le j \le d. \tag{2.6}$$

First we consider the case when $\mathrm{Ker}P$ has a basis of the form

$$f_k(x) = \sum_{i=1}^{dN} a_{ki} x^{\gamma_i}, \quad 0 \le k \le n-1, \tag{2.7}$$

i.e. there are no logarithms. Definition 1.5 in this case is equivalent to

$$\gamma_i - \gamma_j \in N\mathbb{Z} \setminus 0 \quad \text{if } a_{ki}a_{kj} \ne 0, i \ne j. \tag{2.8}$$

We say that the element $f_k(x)$ of the above basis of $\mathrm{Ker}P$ is *associated* to β_s ($1 \le s \le N$) iff

$$\gamma_i - \beta_s \in N\mathbb{Z}_{\ge 0} \text{ if } a_{ki} \ne 0. \tag{2.9}$$

Then up to a relabeling we can take a subset $\{\beta_s\}_{1 \le s \le M}$ such that

$$\beta_s - \beta_t \notin N\mathbb{Z} \quad \text{for} \quad 1 \le s \ne t \le M \tag{2.10}$$

and each element of the basis (2.7) of $\mathrm{Ker}P$ is associated to some β_s from this set. Denote by n_s the number of elements associated to β_s and set $n_s = 0$ for $s > M$. Then $n_1 + \cdots + n_N = n$. We put

$$\beta' = (\beta_1 + n_1 N - n, \beta_2 + n_2 N - n, \ldots, \beta_N + n_N N - n). \tag{2.11}$$

Theorem 2.6 *Let W be a monomial Darboux transformation of the Bessel plane V_β with* $\mathrm{Ker}P$ *satisfying* (2.7, 2.8) *and β' be as above. Then the tau-function τ_W of W lies in the $W_{1+\infty}(N)$-module $\mathcal{M}_{\beta'}$.*

Proof. We shall use the following formula

$$\begin{aligned}\Psi_W(x,z) &= \frac{Wr(f_0(x),\dots,f_{n-1}(x),\Psi_\beta(x,z))}{z^n Wr(f_0(x),\dots,f_{n-1}(x))}\\ &= \frac{\sum \det A^I Wr(x^{\gamma_I})\Psi_I(x,z)}{\sum \det A^I Wr(x^{\gamma_I})},\end{aligned} \tag{2.12}$$

where Wr denotes the Wronski determinant. The sum is taken over all n-element subsets $I=\{i_0<i_1<\dots<i_{n-1}\}\subset\{0,1,\dots,dN-1\}$, $x^{\gamma_I}=\{x^{\gamma_i}\}_{i\in I}$, $A^I=(a_{k,i_l})_{0\le k,\,l\le n-1}$ and $\Psi_I(x,z)$ is the wave function of the above type of Darboux transformations with $f_k(x)=x^{\gamma_{i_k}}$, i.e.

$$\Psi_I(x,z)=z^{-n}L_{\gamma_I}(x,\partial_x)\Psi_\beta(x,z), \qquad \gamma_I=(\gamma_{i_0},\dots,\gamma_{i_{n-1}}).$$

It is important also that

Lemma 2.7 [6] *$\Psi_I(x,z)$ is again a Bessel wave function*:

$$\Psi_I(x,z)=\Psi_{\gamma+dN\delta_I-n\delta}(x,z), \tag{2.13}$$

where the vectors δ_I, δ are defined by

$$(\delta_I)_i=\begin{cases}1, & \textit{if } i\in I\\ 0, & \textit{if } i\notin I\end{cases}$$

and

$$\delta_i=1 \ \text{ for all } \ i\in\{1,\dots,dN\}.$$

We set

$$I_0:=\{1,\dots,n_1,d+1,\dots,d+n_2,\dots,(N-1)d+1,\dots,(N-1)d+n_N\}. \tag{2.14}$$

Then (see (2.6))

$$\gamma_{I_0}=\{\beta_1,\beta_1+N,\dots,\beta_1+(n_1-1)N,\dots,\beta_N,\beta_N+N,\dots,\beta_N+(n_N-1)N\}$$

and clearly

$$\tau_{I_0}=\tau_{\beta'} \tag{2.15}$$

(recall that $\tau_\beta=\tau_\gamma$ when $L_\beta^d=L_\gamma$).

First we shall consider the case when $\det A^{I_0}\neq 0$. Without loss of generality we can put $\det A^{I_0}=1$. Let A_0 be the $n\times dN$ matrix $(a_{ki}\delta_{i,i_k^0})_{0\le k\le n-1,\,1\le i\le dN}$ where $I_0=\{i_0^0<i_1^0<\dots<i_{n-1}^0\}$ is from (2.14). For $\zeta\in\mathbb{C}$ we define the matrix $A(\zeta)$ as follows

$$A(\zeta)=\zeta A+(1-\zeta)A_0. \tag{2.16}$$

Then $A(\zeta)_{ki} = a_{ki}$ for $i = i_k^0$ and $= \zeta a_{ki}$ for $i \neq i_k^0$. Thus (2.8) holds with $A(\zeta)_{ki}$ instead of a_{ki} and the Darboux transformation $W(\zeta)$ of V_β with a matrix $A(\zeta)$ is monomial:

$$W(\zeta) \in Gr_{MB}(\beta).$$

The main idea of the proof of Theorem 2.6 is to consider $W(\zeta)$ as a deformation of $W(0) = V_{\beta'}$. We shall prove that $\tau_{W(\zeta)} \in \mathcal{M}_{\beta'}$ for all ζ, hence $\tau_W = \tau_{W(1)} \in \mathcal{M}_{\beta'}$. We first need a lemma expressing Ψ_I in terms of $\Psi_{\beta'} \equiv \Psi_{I_0}$.

Lemma 2.8 *If* $\det A(\zeta)^I \neq 0$ *for some* ζ *then*

$$\Psi_I(x,z) = x^{-q_I} Q_I(z, \partial_z) \Psi_{I_0}(x,z), \tag{2.17}$$

where Q_I *is a Bessel operator of order* q_I*, divisible by* N *and satisfying*

$$q_I \leq p_I := \sum_{i \in I} \gamma_i - \sum_{i \in I_0} \gamma_i. \tag{2.18}$$

The number p_I *is also divisible by* N*.*

Proof. For $1 \leq s \leq M$ we set $I_s = \{i \in I \mid \gamma_i - \beta_s \in N\mathbb{Z}_{\geq 0}\}$. Then (2.9, 2.10) imply that

$$I = \bigcup_{s=1}^{M} I_s, \quad I_s \cap I_t = \emptyset \quad \text{for } s \neq t \text{ and } \operatorname{card} I_s = n_s.$$

Let

$$\gamma_{I_s} = \left\{ \beta_s + \nu_1^{(s)} N, \beta_s + (\nu_2^{(s)} + 1)N, \ldots, \beta_s + (\nu_{n_s}^{(s)} + n_s - 1)N \right\}$$

for $1 \leq s \leq M$ and $\gamma_{I_s} = \emptyset$ for $s > M$, where $\nu_i^{(s)} \in \mathbb{Z}$, $0 = \nu_1^{(s)} = \cdots = \nu_{k_s}^{(s)} < \nu_{k_s+1}^{(s)} \leq \ldots \leq \nu_{n_s}^{(s)}$. Then

$$p_I = \sum_{s=1}^{M} \sum_{i=1}^{n_s} N \nu_i^{(s)} \geq N \sum_{s=1}^{M} (n_s - k_s) = N(n-k),$$

where $k = \sum_{s=1}^{M} k_s$; we set $k_s = 0$ for $s > M$. We take $q_I = N(n-k)$ and $Q_I = L_\alpha$ be the Bessel operator of order q_I such that

$$L_\alpha L_{\gamma_{I_0}} = L_{\gamma_I} (L_\beta)^{n-k}. \tag{2.19}$$

We shall prove that such L_α exists. Using Lemma 2.5 the right hand side of (2.19) can be represented as

$$L_{\gamma_I} (L_\beta)^{n-k} = L_{\alpha'},$$

where

$$\begin{aligned} \alpha' &= (\gamma_I + (n-k)N\delta_I) \cup \beta^{n-k} \\ &= \bigcup_{s=1}^{N} \Big\{ (\gamma_{I_s} + (n-k)N\delta_{I_s}) \cup (\beta_s, \beta_s + N, \ldots, \beta_s + (n-k-1)N) \Big\}. \end{aligned}$$

We see that α' includes

$$\beta_s, \beta_s + N, \ldots, \beta_s + (n-k+k_s-1)N, \quad (1 \le s \le N)$$

and therefore includes γ_{I_0}, which proves (2.19). Now the proof of (2.17) is straightforward. Using that

$$\Psi_I = x^{-n} L_{\gamma_I} \Psi_\beta, \quad \Psi_{I_0} = x^{-n} L_{\gamma_{I_0}} \Psi_\beta$$

(the Bessel operators act in the variable z), we compute

$$\begin{aligned} x^{-q_I} Q_I \Psi_{I_0} &= x^{-(n-k)N} L_\alpha x^{-n} L_{\gamma_{I_0}} \Psi_\beta = x^{-(n-k)N-n} L_\gamma (L_\beta)^{n-k} \Psi_\beta \\ &= x^{-(n-k)N-n} L_\gamma x^{(n-k)N} \Psi_\beta = x^{-n} L_\gamma \Psi_\beta = \Psi_I. \end{aligned}$$

□

Now we can apply formula (2.12). Obviously

$$Wr(x^{\gamma_I}) = \Delta_I x^{\sum_{i \in I} \gamma_i - \frac{n(n-1)}{2}}, \tag{2.20}$$

where for $I = \{i_0 < \ldots < i_{n-1}\}$ we set

$$\Delta_I = \prod_{r<s} (\gamma_{i_r} - \gamma_{i_s}). \tag{2.21}$$

Using this and (2.17) we write $\Psi_{W(\zeta)}$ as

$$\Psi_{W(\zeta)}(x,z) = \frac{\sum \det A(\zeta)^I \Delta_I x^{p_I - q_I} Q_I(z, \partial_z) \Psi_{I_0}(x,z)}{\sum \det A(\zeta)^I \Delta_I x^{p_I}}.$$

We expand the denominator around $\zeta = 0$. Using that $\Psi_{I_0} = \Psi_{\beta'}$, that p_I and q_I are divisible by N and (1.10) for β' we obtain

$$\Psi_{W(\zeta)}(x,z) = \sum_{i \ge 0} \zeta^i P_i(z, \partial_z) \Psi_{\beta'}(x,z) \tag{2.22}$$

for some operators P_i without constant term for $i \ge 1$ and with $P_0 \equiv 1$. Indeed (2.16) implies that $A(\zeta)^{I_0} = A^{I_0}$ and $\det A(\zeta)^{I_0} = 1$ does not depend on ζ. Therefore for $i \ge 1$ P_i is a linear combination of operators

$$Q_I (L_{\beta'})^{(kp_I - q_I)/N}, \quad k \ge 1, \ I \ne I_0$$

which are nontrivial Bessel operators (see Lemma 2.8) and thus do not have a constant term. For $I = I_0$ $Q_{I_0} \equiv 1$ and $p_I = q_I$, now $\det A(\zeta)^{I_0} = 1$ implies $P_0 \equiv 1$. Denoting $w_{-k} = \partial_x^k \Psi_{\beta'}(x,z)|_{x=1}$ we obtain (see (1.2, A.3))

$$\begin{aligned} \tau_{W(\zeta)} &= \sigma\Big\{ \big(\textstyle\sum \zeta^i P_i w_0\big) \wedge \big(\textstyle\sum \zeta^i P_i w_{-1}\big) \wedge \ldots \Big\} \\ &= \sigma\Big\{ w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots + \zeta(P_1 w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots \\ &\qquad + w_0 \wedge P_1 w_{-1} \wedge w_{-2} \wedge \ldots + \cdots) + \cdots \Big\} \\ &= \tau_{\beta'} + \zeta r(P_1) \tau_{\beta'} + \zeta^2 (r(P_2) + \tfrac{1}{2} r(P_1)^2 - \tfrac{1}{2} r(P_1^2)) \tau_{\beta'} + \cdots \end{aligned}$$

We see that all coefficients at the powers of ζ are polynomials in $r(P_i^k)$ applied to $\tau_{\beta'}$ and thus belong to the $W_{1+\infty}(N)$-module $\mathcal{M}_{\beta'}$ (see (A.12)). Now we shall use the formula

$$\tau_{W(\zeta)} = \frac{\sum \det A(\zeta)^I \Delta_I \tau_I}{\sum \det A(\zeta)^I \Delta_I} \tag{2.23}$$

(see [5]). Because the numerator depends polynomially on ζ the above considerations show that it belongs to $\mathcal{M}_{\beta'}$. Setting $\zeta = 1$ we obtain $\tau_W \in \mathcal{M}_{\beta'}$. This completes the proof of Theorem 2.6 in the case when $\det A^{I_0} \neq 0$.

The general case can be deduced again from the fact that the numerator of (2.23) is polynomial in the entries of A. Up to a relabeling one can suppose that the first n_1 functions of the basis (2.7) of Ker$P$ are associated to β_1, the next n_2 to β_2, etc. Then the Darboux transformation with a matrix

$$A(\xi) = A + \xi E_0, \text{where } E_0 = (\delta_{i\, i_k^0})_{1\leq i\leq dN,\, 0\leq k\leq n-1}$$

is monomial (see (2.8)). Obviously $\det A^{I_0} = \det(A^{I_0} + \xi E) \neq 0$ for all but a finite number of $\xi \in \mathbb{C}$ (where E is the identity matrix) and for them $\tau_{W(\xi)} \in \mathcal{M}_{\beta'}$. Because the numerator of (2.23) (with ζ replaced with ξ) is a polynomial in ξ, it belongs to $\mathcal{M}_{\beta'}$ for all $\xi \in \mathbb{C}$ and for $\xi = 0$ it is exactly τ_W (recall that a tau-function is defined up to a multiplication by a constant). $\square$

2.3

Now we shall consider the general case of a monomial Darboux transformation of V_β, $\beta \in \mathbb{C}^N$. Using repeatedly Lemma 2.7 with $n = d = 1$ we see that $V_\beta \in Gr_{MB}(\nu)$ with ν of the form

$$\nu = (\underbrace{\nu_1, \nu_1, \ldots, \nu_1}_{N_1}, \ldots, \underbrace{\nu_p, \nu_p, \ldots, \nu_p}_{N_p}) \tag{2.24}$$

such that

$$\nu_i - \nu_j \notin N\mathbb{Z} \quad \text{for } i \neq j \tag{2.25}$$

$(N_1 + \cdots + N_p = N)$.

Let $W \in Gr_{MB}(\nu)$, i.e. Ker$P$ has a basis consisting of several groups of the form described in Definition 1.5:

$$\sum_{j=l}^{j_0} l! \binom{j}{l} f_j(x) (\ln x)^{j-l}, \quad 0 \leq l \leq j_0 \tag{2.26}$$

where $j_0 \leq N_i - 1$ and

$$f_j(x) = \sum_{k=0}^{d-1} b_{kj} x^{\nu_i + kN}. \tag{2.27}$$

We say that the element (2.26) of Ker$P$ has *level* $j_0 - l$. For $1 \leq s \leq p$ we denote by n_s^r the number of elements in the basis of Ker$P$ of level r associated to ν_s (see (2.9)). Put

$$\nu' = (\nu_1 + n_1^0 N - n, \ldots, \nu_1 + n_1^{N_1 - 1} N - n, \ldots, \nu_p + n_p^0 N - n, \ldots, \nu_p + n_p^{N_p - 1} N - n). \tag{2.28}$$

Theorem 2.9 *If W is a monomial Darboux transformation of V_ν with ν, ν' and* Ker$P$ *as above then $\tau_W \in \mathcal{M}_{\nu'}$.*

Proof. We shall make a limit in Theorem 2.6. We note that since $(\ln x)^j = \partial_\lambda^j x^\lambda|_{\lambda=0}$ we have

$$(\ln x)^j = \lim_{\epsilon\to 0} \epsilon^{-j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} x^{-\epsilon k}.$$

Set $\nu(\epsilon) = (\nu_1, \nu_1+\epsilon, \ldots, \nu_1+(N_1-1)\epsilon, \ldots, \nu_p, \nu_p+\epsilon, \ldots, \nu_p+(N_p-1)\epsilon)$. Consider the Darboux transformation $W(\epsilon)$ of $V_{\nu(\epsilon)}$ with a basis of Ker$P(\epsilon)$ consisting of groups of the form (cf. (2.26, 2.27))

$$g_l(x) = x^{\epsilon(j_0-l)} \sum_{j=l}^{j_0} l! \binom{j}{l} f_j(x) \sum_{k=0}^{j-l} \epsilon^{l-j} (-1)^k \binom{j-l}{k} x^{-\epsilon k}, \quad 0 \le l \le j_0.$$

We shall show that this transformation is monomial. More precisely, we shall prove that Ker$P(\epsilon)$ has a basis consisting of groups of elements of the form

$$h_l(x) = x^{\epsilon(j_0-l)} \sum_{j=l}^{j_0} l! \binom{j}{l} \epsilon^{l-j} f_j(x), \quad 0 \le l \le j_0.$$

This is an obvious consequence of the identity

$$g_l(x) = \sum_{k=0}^{j_0-l} \frac{(-1)^k \epsilon^{-k}}{k!} h_{l+k}(x), \quad 0 \le l \le j_0.$$

We apply Theorem 2.6 for $W(\epsilon)$ noting that exactly n_s^r elements of the above basis of Ker$P(\epsilon)$ are associated to $\nu_s + r\epsilon$. Taking the limit $\epsilon \to 0$ completes the proof. $\square$

2.4

As an illustration to Theorem 2.6 we shall consider the case $n = d = 1$. Now the matrix A is $1 \times N$:

$$A = (a_1 \ a_2 \ \ldots \ a_N),$$

subsets I consist of one element: $I = \{i\}$, and

$$\Psi_I \equiv \Psi_i = \frac{1}{z}\left(\partial_x - \frac{\beta_i}{x}\right)\Psi_\beta = \Psi_{(\beta_1-1,\ldots,\beta_i+N-1,\ldots,\beta_N-1)}. \tag{2.29}$$

The formula (2.12) now becomes

$$\Psi_W(x,z) = \frac{1}{zx}\left(x\partial_x - \frac{\sum a_i\beta_i x^{\beta_i}}{\sum a_i x^{\beta_i}}\right)\Psi_\beta(x,z). \tag{2.30}$$

Let $a_1 \neq 0$. The Darboux transformation is monomial when

$$\beta_i - \beta_1 = N\alpha_i, \quad \alpha_i \in \mathbb{Z} \quad \text{for} \quad a_i \neq 0$$

and up to a relabeling we can suppose that all α_i are positive. Then $I_0 = \{1\}$, $\Psi_{I_0} \equiv \Psi_1 \equiv \Psi_{\beta'}$, where $\beta' = (\beta_1 + N - 1, \beta_2 - 1, \ldots, \beta_N - 1)$. Using that

$$\begin{aligned}\Psi_\beta(x,z) &= x^{-N} L_\beta(z,\partial_z)\Psi_\beta(x,z) = (xz)^{-N} P_\beta(D_z)\Psi_\beta(x,z) \\ &= (xz)^{-N+1}\frac{P_\beta(D_z+1)}{D_z+1-\beta_1}\Psi_{\beta'}(x,z)\end{aligned}$$

and that

$$P_\beta(D_z) = \frac{D_z - \beta_1}{D_z - (\beta_1 + N)} P_{\beta'}(D_z - 1), \quad P_{\beta'}(D_z) = \frac{D_z - (\beta_1 + N - 1)}{D_z - (\beta_1 - 1)} P_\beta(D_z + 1) \tag{2.31}$$

we obtain from (2.30)

$$\Psi_W(x,z) = \frac{1}{\sum a_i x^{N\alpha_i}} P_1(z,\partial_z)\Psi_{\beta'}(x,z), \tag{2.32}$$

where

$$P_1(z,\partial_z) = \sum_{i=1}^{N} a_i \left\{ -N\alpha_i z^{-N} \frac{P_{\beta'}(D_z)}{D_z - \beta'_1} \left(z^{-N} P_{\beta'}(D_z)\right)^{\alpha_i} \right\}. \tag{2.33}$$

Then $A(\zeta) = (a_1 \ \zeta a_2 \ \ldots \ \zeta a_N)$, the numerator of (2.23) is equal to $\tau_{\beta'} + \zeta r(P_1)\tau_{\beta'}$ and up to a constant

$$\tau_W = r(P_1)\tau_{\beta'}. \tag{2.34}$$

Example 2.10 Let $A = (0 \ 1 \ 0 \ \ldots \ 0)$ and $\beta'_2 - \beta'_1 = N\alpha = N(\alpha_2 - 1)$, $\alpha \in \mathbb{Z}_{\geq 0}$. Set

$$\beta'' = (\beta'_1 - N, \beta'_2 + N, \beta'_3, \ldots, \beta'_N) = (\beta_1 - 1, \beta_2 + N - 1, \beta_3 - 1, \ldots, \beta_N - 1).$$

Then the module $\mathcal{M}_{\beta''}$ embeds in $\mathcal{M}_{\beta'}$. The singular vector $\tau_{\beta''}$ is given by (2.34, 2.33). Up to some changes of notation ($\beta_i = Nr_i$, etc.) in this way we recover Theorem 6 from [4]. $\square$

Example 2.11 Let $N = 2$, $\beta_1 - \beta_2 = 2\alpha$, $\alpha \in \mathbb{Z}_{\geq 0}$. Then the tau-function $\tau_\alpha := \tau_{(1/2+\alpha, 1/2-\alpha)}$ is highest weight vector for the reducible $W_{1+\infty}(2)$-module $\mathcal{M}_{(1/2+\alpha,1/2-\alpha)}$, which will be denoted below by $\mathcal{M}_\alpha$. Example 2.10 gives that $\mathcal{M}_0 \supset \mathcal{M}_2 \supset \mathcal{M}_4 \supset \ldots$ and $\mathcal{M}_1 \supset \mathcal{M}_3 \supset \mathcal{M}_5 \supset \ldots$. Any bispectral tau-function corresponding to an "even" potential [9] can be obtained by a monomial Darboux transformation with $d = n \leq \alpha$ from τ_α as shown in [17] (see also [6], Example 5.3). Theorem 2.6 shows that $\tau_W \in \mathcal{M}_{\alpha-n}$. On the other hand $\tau_W \in Gr^{(2)}$, which gives that τ_W belongs to the Vir-module $M^\infty_{\alpha-n}$ introduced in [13] (see also [10]). The modules M^∞_α, $\alpha \in \mathbb{Z}_{\geq 0}$ are shown to be the reducible Verma modules over Vir with $c = 1$ (whose highest weight vectors are the above tau-functions τ_α). In this way we obtain:

Corollary 2.12 *Any tau-function τ_W of an "even" potential can be obtained by a monomial Darboux transformation from the highest weight vector τ_α of a reducible Vir-module M_α^∞, $\alpha \in \mathbb{Z}_{\geq 0}$ (defined in* [13]*) and it belongs to the module $M_{\alpha-n}^\infty$, where n is the order of the operator P ($n \leq \alpha$). Conversely, any tau-function in M_α^∞ is tau-function of an "even" potential* [9].

Consider the set of modules $\mathcal{M}_\beta$ of the most degenerate case $\beta_i - \beta_j \in N\mathbb{Z}$ for all i, j ($\beta \in \mathbb{C}^N$). The embeddings among these modules are described by N lattices: the k-th one of them having a maximal module $\mathcal{M}_{\beta^{(k)}}$,

$$\beta^{(k)} = (\underbrace{b+N, \ldots, b+N}_{k}, \underbrace{b, \ldots, b}_{N-k}),\ b = \frac{N-2k-1}{2},$$

for $0 \leq k \leq N-1$ (cf. Example 2.10). Lemma 2.7 implies that the set of monomial Darboux transformations of $\beta^{(k)}$ with ord$P \in N\mathbb{Z}$ coincides with the set of monomial Darboux transformations of $\beta^{(0)}$ with ord$P \in k + N\mathbb{Z}$. In the latter case the corresponding $\mathcal{M}_{\nu'}$ given by Theorem 2.9 belongs to the k-th lattice, i.e. it is a submodule of $\mathcal{M}_{\beta^{(k)}}$. So we obtain the following corollary.

Corollary 2.13 *The manifold of monomial Darboux transformations from $\beta^{(k)}$ with* ord$P \in N\mathbb{Z}$ *coincides with the manifold of tau-functions lying in the module $\mathcal{M}_{\beta^{(k)}}$.*

Remark 2.14 We shall consider another aspect of Theorem 2.6. Recall that the subalgebras $W_{1+\infty}(d)$, $d \in \mathbb{N}$ of $W_{1+\infty}$ are isomorphic to $W_{1+\infty} \equiv W_{1+\infty}(1)$ and a representation of $W_{1+\infty}(d)$ with central charge N gives rise to a representation of $W_{1+\infty}$ with central charge dN. Each singular vector of $W_{1+\infty}$ is obviously a singular vector of $W_{1+\infty}(d)$ but the converse is not always true. It is an intersting question to describe the latter.

In our terminology this question can be reformulated as follows. *Which Bessel tau-functions τ_α, $\alpha \in \mathbb{C}^{dN}$ lie in a $W_{1+\infty}(N)$-module $\mathcal{M}_\beta$, for some $\beta \in \mathbb{C}^N$?* Such tau-functions are given by Theorem 2.6: if I is an n-element subset of $\{1, \ldots, dN\}$ then $\tau_{\beta^d + dN\delta_I - n\delta} \in \mathcal{M}_{\beta'}$ (see (2.11) and Lemma 2.7).

Let us consider the simplest case $N = 1$ and set $d = 2n$, $I = \{1, 3, \ldots, 2n-1\}$. Then $\beta = (0)$, $\beta^d = (0, 1, \ldots, 2n-1)$ and

$$\beta^d + dN\delta_I - n\delta = (1-n, 3-n, \ldots, n-1) \cup (n, n+2, \ldots, 3n-2) = (1-n, n)^n.$$

So we obtain that $\tau_{(1-n,n)}$ lies in $W_{1+\infty}(1)$ module $\mathcal{M}_{(0)} = \mathbb{C}[t_1, t_2, \ldots]$, i.e. they are polynomials of $t_1, t_2, \ldots$ (obviously it coincides with the module over the Heisenberg algebra with highest weight vector $\tau_{(0)} = 1$). These tau-functions play an important role in the "KdV case" of [9] and are connected with the rank 1 bispectral algebras which contain an operator of order 2. The general case of rank N bispectral algebras containing an operator of order dN motivates the study of the above considered "embeddings".

3 Hierarchies of symmetries of the manifolds of monomial Darboux transformations

An immediate consequence of the results of the previous section is the existence of hierarchies of symmetries preserving the manifolds of monomial Darboux transformations.

Denote by $W^+_{1+\infty}(N)$ the subalgebra of $W_{1+\infty}(N)$ spanned by J^l_{Nk}, $k, l \geq 0$.

Theorem 3.1 *For $\beta \in \mathbb{C}^N$ the vector fields corresponding to $W^+_{1+\infty}(N)$ are tangent to the manifold $Gr_{MB}(\beta)$ of monomial Darboux transformations. More precisely, if $W \in Gr_{MB}(\beta)$ then*

$$\exp\left(\sum_{i=1}^{p} \lambda_i J^{l_i}_{Nk_i}\right) \tau_W \tag{3.1}$$

is a tau-function associated to a plane from $Gr_{MB}(\beta)$ for arbitrary $p \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $l_i, k_i \in \mathbb{Z}_{\geq 0}$.

Proof. Indeed, if $W \in Gr_{MB}(\beta)$, $\beta \in \mathbb{C}^N$ then $\tau_W \in \mathcal{M}_{\beta'}$ for some $\beta' \in \mathbb{C}^N$ such that $V_{\beta'} \in Gr_{MB}(\beta)$. Now from the gradation of $W_{1+\infty}(N)$ it is clear that $\sum_{i=1}^{p} \lambda_i J^{l_i}_{Nk_i}$ acts nilpotently on τ_W, i.e. (3.1) is well-defined and also belongs to the module $\mathcal{M}_{\beta'}$. Moreover, it is a tau-function (see [1]) and Theorem 2.1 shows that its corresponding plane belongs to $Gr_{MB}(\beta') = Gr_{MB}(\beta)$. □

Let us introduce the following terminology. A $\beta \in \mathbb{C}^N$ is called generic if V_β cannot be obtained by a Darboux transformation of some V_α, $\alpha \in \mathbb{C}^M$, $M < N$. We put $Gr^{(N)}_{MB} = \bigcup_\beta Gr_{MB}(\beta)$, $\beta \in \mathbb{C}^N$–generic. These manifolds are important because they give bispectral algebras of rank N [6]. Therefore Theorem 3.1 implies:

Corollary 3.2 *The manifold $Gr^{(N)}_{MB}$ of monomial Darboux transformations which give bispectral algebras of rank N is preserved by the vector fields corresponding to $W^+_{1+\infty}(N)$.*

An interesting question is when a polynomial Darboux transformation of a Bessel operator L_β of order N gives again an operator of order N (see [9] for $N = 2$ and [6]). In [6], Proposition 5.4 we proved that for generic β such transformation is necessarilly monomial. The corresponding manifold $Gr_{MB}(\beta) \cap Gr^{(N)}$ is also preserved by an hierarchy of symmetries. More precisely, in the bosonic realization (1.21) of $W_{1+\infty}(N)$ we put $J_{kN} = 0$, $k \in \mathbb{Z}$ and define

$$\overline{L}_m = \frac{1}{N} J^1_{mN}|_{J_{kN}=0,\, k\in\mathbb{Z}} = \frac{1}{N} \sum_{i\in\mathbb{Z}\setminus N\mathbb{Z}} {:}J_{mN-i}J_i{:}.$$

The operators $\overline{L}_m$, $m \in \mathbb{Z}$ form a Virasoro algebra with central charge $N - 1$ which we denote by Vir_N. Denote by Vir^+_N the subalgebra spanned by $\overline{L}_m$, $m \geq 0$. Then we can formulate the following theorem which for $N = 2$ contains Magri–Zubelli's result [17].

Theorem 3.3 *The manifold $Gr_{MB}(\beta)\cap Gr^{(N)}$ is preserved by the vector fields corresponding to Vir_N^+. More precisely, if $W\in Gr_{MB}(\beta)\cap Gr^{(N)}$ then*

$$\exp\left(\sum_{i=1}^{p}\lambda_i\overline{L}_{k_i}\right)\tau_W \tag{3.2}$$

is a tau-function associated to a plane from $Gr_{MB}(\beta)\cap Gr^{(N)}$ for arbitrary $p\in\mathbb{N}$, $\lambda_i\in\mathbb{C}$, $k_i\geq 0$.

Proof. Obviously formula (3.2) gives exactly the same result as

$$\exp\left(\sum\lambda_i N^{-1}J^1_{k_iN}\right)\tau_W.$$

This is because $\tau_W(t)$ does not depend on t_{kN} and in $J^1_{k_iN}$, $k_i\geq 0$ the variables t_{kN} are present only as coefficients of differentiations with respect to t_{mN}. This implies that (3.2) is a tau-function of a plane belonging to $Gr^{(N)}$. □

Remark 3.4 The manifold $Gr_{MB}(\beta)\cap Gr^{(dN)}$, $\beta\in\mathbb{C}^N$ is preserved by the vector fields corresponding to the subalgebra of $W^+_{1+\infty}(N)$ generated by

$$\begin{aligned} &J_{kN}, \quad \text{for } d\not| k,\ k\geq 0;\\ &\widetilde{L}_m=\sum_{i\in\mathbb{Z}\setminus\ dN\mathbb{Z}} :J_{mN-i}J_i:, \quad \text{for } m\geq 0. \end{aligned}$$

(Because on $Gr^{(dN)}$ $\widetilde{L}_m$ acts as J^1_{mN}.)

For $N=1$, $d=2$ we recover the well known fact that the potentials from the "KdV case" of [9] are preserved by the KdV flows.

For generic $\beta\in\mathbb{C}^N$ $Gr_{MB}(\beta)\cap Gr^{(dN)}$ gives bispectral algebras of rank N containing an operator of order dN. However for $d>1$, $N>1$ these are not all such algebras, cf. [6]. It is still an open problem to describe the symmetries of the latter. □

We shall conclude this section with some comments. As $Gr^{(N)}$ is a reduction of Gr, the (associative) algebra W_N is a reduction of $W_{1+\infty}(N)$ – see e.g. [12, 20]. In more details, the fields $J^0(z), J^1(z),\dots,J^{N-1}(z)$ generate the (vertex operator) algebra $\mathcal{W}(gl_N)$ [11]. Its reduction $\mathcal{W}(sl_N)$ is obtained by putting $J_{kN}=0$, $k\in\mathbb{Z}$ in (1.21); the modes of the corresponding fields generate the so-called W_N algebra. (More precisely, this is a representation of W_N with $c=N-1$.) Then $Vir_N\subset W_N$ and we conjecture that Theorem 3.3 is valid with W_N^+ instead of Vir_N^+.

Appendix

In this appendix we give the technical proofs of some of the results from Sect. 1 and explain them in more details.

First, following [16], we shall recall the boson-fermion correspondence. Recall the definition of the fermionic Fock space F from subsection 1.1. The free fermions can be realized as wedging and contracting operators:

$$\psi_{-j+\frac{1}{2}}\left(v_{i_0}\wedge v_{i_1}\wedge\ldots\right)=v_j\wedge v_{i_0}\wedge v_{i_1}\wedge\ldots$$
$$\psi^*_{j-\frac{1}{2}}\left(v_j\wedge v_{i_0}\wedge v_{i_1}\ldots\right)=v_{i_0}\wedge v_{i_1}\wedge\ldots$$

They satisfy the canonical anticommutation relations

$$\left[\psi_\lambda,\psi^*_\mu\right]_+=\delta_{\lambda,-\mu},\quad \left[\psi_\lambda,\psi_\mu\right]_+=0,\quad \left[\psi^*_\lambda,\psi^*_\mu\right]_+=0, \tag{A.1}$$

where $[a,b]_+=ab+ba$.

Let gl_∞ be the Lie algebra of all $\mathbb{Z}\times\mathbb{Z}$ matrices having only a finite number of non-zero entries. One can define a representation r of gl_∞ in the fermionic Fock space F as follows. For the basis $E_{ij}\in gl_\infty$ put

$$r\left(E_{ij}\right)=\psi_{-i+\frac{1}{2}}\psi^*_{j-\frac{1}{2}} \tag{A.2}$$

and continue this by linearity. Then for $A\in gl_\infty$

$$r(A)(w_0\wedge w_{-1}\wedge w_{-2}\wedge\ldots)=Aw_0\wedge w_{-1}\wedge w_{-2}\wedge\ldots+w_0\wedge Aw_{-1}\wedge w_{-2}\wedge\ldots+\cdots. \tag{A.3}$$

The above defined representation r obviously cannot be continued on the Lie algebra $\widetilde{gl}_\infty$ of all $\mathbb{Z}\times\mathbb{Z}$ matrices with finite number of non-zero diagonals. If we regularize it by

$$\hat{r}\left(E_{ij}\right)=:\psi_{-i+\frac{1}{2}}\psi^*_{j-\frac{1}{2}}:, \tag{A.4}$$

where as usual $:\psi_\mu\psi^*_\nu:=\psi_\mu\psi^*_\nu$ for $\nu>0$ and $-\psi^*_\nu\psi_\mu$ for $\nu<0$, this will give a representation for the central extension $\widehat{gl}_\infty=\widetilde{gl}_\infty\oplus\mathbb{C}c$ of $\widetilde{gl}_\infty$. Here the central charge c acts as a multiplication by 1. Define the free fermionic fields

$$\psi(z)=\sum_{j\in\mathbb{Z}}\psi_{j-\frac{1}{2}}z^{-j}\quad\text{and}\quad\psi^*(z)=\sum_{j\in\mathbb{Z}}\psi^*_{j-\frac{1}{2}}z^{-j}.$$

Then the anticommutation relation (A.1) can be written as

$$\Big[\psi(z_1),\psi^*(z_2)\Big]_+=\delta(z_{12}), \tag{A.5}$$

where $z_{12}=z_1-z_2$ and $\delta(z_{12})=\sum_{n\in\mathbb{Z}}z_1^nz_2^{-n-1}$. Introduce also the $\hat{u}(1)$ current

$$J(z)=:\psi^*(z)\psi(z):=\sum_{n\in\mathbb{Z}}J_nz^{-n-1}. \tag{A.6}$$

The modes J_n generate the Heisenberg algebra (1.19).

The above introduced spaces $F^{(m)}$ are spaces of irreducible representations of the Heisenberg algebra with charge m and central charge $c=1$. Using that such a representation is unique up to isomorphism we obtain the isomorphism known as the

boson-fermion correspondence (1.1, 1.20). In terms of the states $|m\rangle = v_m \wedge v_{m-1} \wedge \ldots$ and the operator $H(t) = -\sum_{k=0}^{\infty} t_k J_k$ we have for $|\varphi\rangle \in F$

$$\sigma(\,|\varphi\rangle) = \sum_{m\in\mathbb{Z}} \langle m|\; e^{H(t)}\; |\varphi\rangle Q^m. \tag{A.7}$$

We also introduce the scalar bosonic field:

$$\phi(z) = \hat{q} + J_0 \log z + \sum_{n\neq 0} J_n \frac{z^{-n}}{-n} \tag{A.8}$$

with operator product expansion $\phi(z_1)\phi(z_2) \sim \log(z_1 - z_2)$, which is equivalent to (1.19) and

$$\Big[J_n, \hat{q}\Big] = \delta_{n,0}, \tag{A.9}$$

and such that

$$\exp \hat{q} = Q, \quad J(z) = \partial_z \phi(z), \quad Q^m = {:}e^{m\phi(z)}{:}|0\rangle|_{z=0}.$$

Then the fermionic fields $\psi(z)$, $\psi^*(z)$ act on the bosonic Fock space B as

$$\psi^*(z) = {:}e^{\phi(z)}{:}, \qquad \psi(z) = {:}e^{-\phi(z)}{:}. \tag{A.10}$$

Here as usual ${:}J_nJ_m{:} = J_nJ_m$ for $m > n$, ${:}J_nJ_m{:} = J_mJ_n$ for $m < n$ and ${:}\hat{q}J_0{:} = {:}J_0\hat{q}{:} = \hat{q}J_0$.

One can define a natural embedding of $W_{1+\infty}$ in $\widehat{gl}_\infty$ in the following way [14, 15]. Consider a realization of $\mathbb{V}$ as the space of Laurent series in z^{-1}. Fixing the basis $v_j = z^{-j}$ of $\mathbb{V}$ each element $A \in \mathcal{D}$ corresponds to a matrix $\phi_0(A) \in \widetilde{gl}_\infty$, i.e. one defines an embedding $\phi_0 \colon \mathcal{D} \hookrightarrow \widetilde{gl}_\infty$. This embedding can be extended to an embedding

$$\hat{\phi}_0 \colon W_{1+\infty} \hookrightarrow \widehat{gl}_\infty.$$

In the case when c acts as a multiplication by 1 we can obtain, using (A.4), a free field realization of $W(A)$:

$$W(A) = \mathrm{Res}_{z=0} {:}\psi(z) A \psi^*(z){:} \tag{A.11}$$

for $A \in \mathcal{D}$. In the notation of (A.4) this means

$$W(A) = \hat{r}(A). \tag{A.12}$$

Also note that $\hat{r}(A) = r(A)$ for operators A having $\mathrm{diag}\phi_0(A) = 0$.

From (A.11) we derive a bosonic realization of the fields $J^l(z)$:

$$J^l(z) = {:}(\partial_z^l \psi^*(z))\psi(z){:} \tag{A.13}$$

which combined with (A.10) gives

$$J^l(z) = \frac{1}{l+1} {:}e^{-\phi(z)} \partial_z^{l+1} e^{\phi(z)}{:}. \tag{A.14}$$

This and the Taylor formula imply (1.21). Indeed,

$$
\begin{aligned}
\sum_{l\geq 0}\frac{x^{l+1}}{l!}J^l(z) &= :\Big(\sum_{l\geq 0}\frac{x^{l+1}}{(l+1)!}\partial_z^{l+1}e^{\phi(z)}\Big)e^{-\phi(z)}: \\
&= :\Big(e^{\phi(z+x)}-e^{\phi(z)}\Big)e^{-\phi(z)}: = :\Big(e^{\sum_{k\geq 0}\frac{x^k}{k!}\partial^k\phi(z)}e^{-\phi(z)}-1\Big): \\
&= \sum_{l\geq 1}x^l :S_l\Big(\frac{\partial\phi}{1!},\frac{\partial^2\phi}{2!},\dots\Big):.
\end{aligned}
$$

Comparing the coefficients at x^{l+1} and using that $J(z)=\partial_z\phi(z)$ we get (1.21).

To describe the action of the involution a on $W_{1+\infty}$ we introduce another convenient basis V_k^l, c of $W_{1+\infty}$ through the fields

$$
V^l(z)=\sum_{k\in\mathbb{Z}}V_k^l z^{-k-l-1}. \tag{A.15}
$$

These fields are *quasiprimary* of dimension $l+1$ with respect to the Virasoro algebra generated by V_k^1, c and are used essentially by Cappelli, Trugenberger and Zemba in their study of the Quantum Hall Effect (see [7] and references therein). They can be defined by [2]

$$
V^l(z)=\frac{l!}{(2l)!}\partial_1^l(-\partial_2)^l\Big\{z_{12}^l:\psi^*(z_1)\psi(z_2):\Big\}\Big|_{z_1=z_2=z} \tag{A.16}
$$

(for the connection with the fields $J^l(z)$ see [2], eq. (1.41)). We have an analog of (A.14):

$$
V^l(z)=\frac{1}{l}\binom{2l}{l}^{-1}\sum_{k=0}^{l-1}\binom{l}{k}\binom{l}{k+1}\partial_1^{l-k}(-\partial_2)^{k+1}:e^{\phi(z_1)-\phi(z_2)}:\Big|_{z_1=z_2=z}. \tag{A.17}
$$

This leads to an analog of (1.21) proved in the same way:

$$
\begin{aligned}
V^l(z) &= \frac{(l-1)!\,l!\,(l+1)!}{(2l)!}\sum_{k=0}^{l-1}\binom{l}{k}\binom{l}{k+1}\binom{l+1}{k+1}^{-1}(-1)^{k+1} \\
&\times\ :S_{l-k}\Big(\frac{J(z)}{1!},\frac{\partial J(z)}{2!},\dots\Big)S_{k+1}\Big(-\frac{J(z)}{1!},-\frac{\partial J(z)}{2!},\dots\Big):
\end{aligned} \tag{A.18}
$$

Substituting $a(J(z))=J(-z)$ for $J(z)$ in (A.18) it is easy to see that

$$
a(V^l(z))=V^l(-z),\qquad l\geq 0 \tag{A.19}
$$

(we use that the elementary Schur polynomials S_l are homogeneous of degree l if $\deg t_k=k$). In terms of the modes $a(V_k^l)=(-1)^{l+k+1}V_k^l$ showing that $W_{1+\infty}(N)$ is preserved by the involution a.

At the end we shall give the *proof of Lemma 2.3*. Comparing (A.8, A.10) with (1.4) we see that

$$
\psi^*(z)=:e^{\phi(z)}:=QX(t,z).
$$

From (A.13) and (A.5) we derive commutation relations

$$\left[J^l(z_1), \psi^*(z_2)\right] = \lim_{z_3 \to z_1} \partial^l_{z_1} {:}\left[\psi^*(z_1)\psi(z_3), \psi^*(z_2)\right]{:} = \delta(z_{12})\partial^l_{z_2}\psi^*(z_2). \qquad \text{(A.20)}$$

Using that $J_0 Q = Q(J_0 + 1)$ (see (A.9)) we compute

$$Q^{-1}J^l(z) = Q^{-1} \lim_{z_{1,2}\to z} \frac{\partial_1^{l+1}}{l+1} {:}e^{\phi(z_1)-\phi(z_2)}{:} = \lim_{z_{1,2}\to z} \frac{\partial_1^{l+1}}{l+1} {:}e^{\phi(z_1)-\phi(z_2)}{:} \frac{z_1}{z_2} Q^{-1}$$

$$= \begin{cases} (J^l(z) + lz^{-1}J^{l-1}(z))Q^{-1}, & \text{for } l > 0. \\ (J^0(z) + z^{-1})Q^{-1}, & \text{for } l = 0. \end{cases}$$

Then

$$X(t,z_1)J^l(z_2) = Q^{-1}\psi^*(z_1)J^l(z_2) = Q^{-1}J^l(z_2)\psi^*(z_1) - Q^{-1}\delta(z_{12})\partial_1^l\psi^*(z_1)$$

$$= \left(J^l(z_2) + z_2^{-1} l J^{l-1}(z_2)\right) X(t,z_1) - \delta(z_{12})\partial_1^l X(t,z_1)$$

(for $l = 0$ instead of $lJ^{l-1}(z_2)$ put 1). Comparing the coefficients of z_2^{-k-l-1} in both sides completes the proof.

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